Relations

- Relations: Definition and Notation
- Properties of Relations
- Combining Relations
- Operations on Relations: Projection and Join
- Equivalence Relations and Equivalence Classes
- Partial Order

Imdad ullah Khan

Partial Orders: Motivation

Consider the following courses taken by CS students

- CS100: Introduction to Computing
- CS202: Data Structures
- CS210: Discrete Mathematics
- CS310: Algorithms
- CS341: Database Systems
- CS381: Operating Systems

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Order and dependencies among courses

- CS100 must be taken before all other courses
- CS202 and CS210 must be taken before CS310
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Such a system can be modeled using a relation called **partial order**

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For example pre-requisite order to courses

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Pronounced as a **preceeds** b

Do not confuse $\preccurlyeq$ with $\leq$ $\quad$ $\preccurlyeq$ denotes partial ordering

Partially Ordered Set (Poset)

Poset (Partially Ordered Set) (S, R)

A set S together with a partial order R, (S, R) is called a *poset*

Partial Order: Example

$(\geq)$ relation is a partial ordering on the set of integers $\mathbb{Z}$

$$(\geq) = \{\cdots (1,1),(1,0),(1,-2),(0,-5),(-5,-6),(3,2),(3,1),(3,-9)\cdots\}$$

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- $a \geq a$ for every integer a $\triangleright \geq$ is reflexive
- If $a \geq b$ and $b \geq a$, then $a = b$ $\triangleright \geq$ is antisymmetric
- If $a \geq b$ and $b \geq c$, then $a \geq c$ $\triangleright \geq$ is transitive

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ICP 6-33 Show that $(\mathbb{Z}, \leq)$ is a poset.

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- $a|a$ whenever a is a positive integer ▷ reflexive
- if a and $b \in \mathbb{Z}^+$ with $a|b$ and $b|a$, then $a = b$ ▷ antisymmetric
- If $a|b$ and $b|c$, then there are positive integers k and ℓ such that $b = ak$ and $c = b\ell$. Hence, $c = a(k\ell)$, so $a|c$ ▷ transitive

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ICP 6-34 Let $S = \{a, b, c\}$. List all ordered pairs in $\subseteq$ relation on $\mathcal{P}(\mathcal{S})$

- $A \subseteq A$ for all subsets A of S $\triangleright$ $\subseteq$ is reflexive
- if $A \subseteq B$ and $B \subseteq A$, then $A = B$ $\triangleright$ $\subseteq$ is antisymmetric
- If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$ $\triangleright$ $\subseteq$ is transitive

ICP 6-35 Let S be a set. Is $(\mathcal{P}(\mathcal{S}), \subset)$ a poset ?

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- If x is older than y, then y cannot be older than x $\triangleright$ R is antisymmetric
- if x is older than y and y is older than z, then x is older than z $\triangleright$ R is transitive
- A person cannot be older than him/herself $\triangleright$ R is **Not Reflexive**

$\therefore$ R is not a partial order

Comparability

The elements a and b of a poset $(S, \preccurlyeq)$ are called *comparable* if either $a \preccurlyeq b$ or $b \preccurlyeq a$, otherwise they are *incomparable*

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ICP 6-37 Are 5 and 7 comparable? ▷ No, because $5 \nmid 7$ and $7 \nmid 5$

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- The relation "$\leq$" on the set of integers is a total order; $(Z, \leq)$ since for every $a, b \in Z$, it must be the case that $a \leq b$ or $b \leq a$
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- What happens if we replace $\leq$ with $<$?

  $(Z, <)$ is not even a poset (non-reflexive) $\therefore$ not a total order

- $(\mathbb{Z}^+, |)$ is not totally ordered because it contains elements that are incomparable, such as 5 and 7

Now it should be clear why partial orders are called "partial"

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  We don't know. $\mathbb{Z}$ is unbounded from below (and above)

Hasse Diagrams

A poset $(S, \preccurlyeq)$ can be represented by a digraph

- Each element $a \in S$ is a node
- if $a \preccurlyeq b$, then (a, b) is an edge

Constructing Hasse diagram:

- Remove all self-loops
- Remove all transitive edges
- Remove directions - assume that the orientations are upwards

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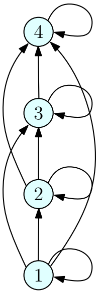
Hasse diagram of $(\{1,2,3,4\}, \leq)$



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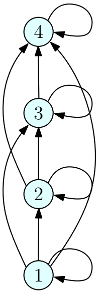

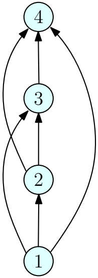

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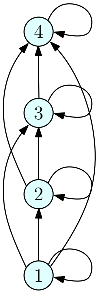

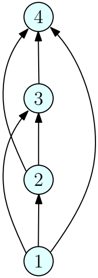

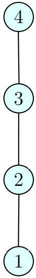

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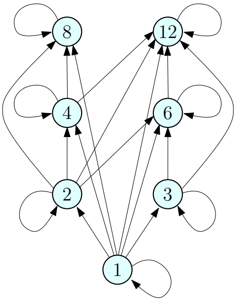
Hasse diagram of $(\{1, 2, 3, 4, 6, 8, 12\}, |)$



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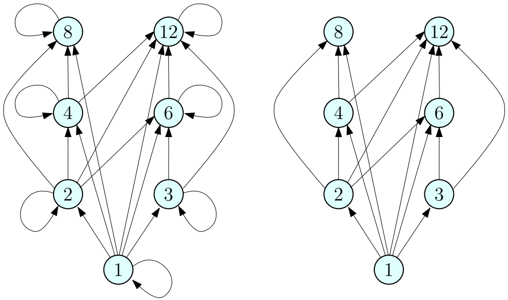

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